Student Notes

Prep Session Topic: Computing Derivatives

It goes without saying that derivatives are an important part of the calculus and you need to be able to compute them. You should know the derivatives of all the functions you've been studying;

- Polynomials
- Rational functions (quotients) and functions with radicals
- Trig functions
- Inverse trig functions (by implicit differentiation)
- Exponential and logarithmic functions

The AP exams will ask you to find derivatives using the various techniques and rules including

- The Power rule for integer, rational (fractional) exponents, expressions with radicals.
- Derivatives of sum, differences, products, and quotients.
- The chain rule for composite functions. Nearly every multiple-choice question on differentiation from the 1997, 1998 and 2003 released exams uses the Chain Rule.
- Implicit differentiation on multiple-choice and often an entire free-response question.
- Fundamental Theorem of Calculus (FTC) – differentiate an integral.
- The derivative of expression in functional form ($f\left(g\left(x\right)\right)$, $f\left(x\right)g\left(x\right)$, etc.) where the functions are not given. Values are taken from a table or graph.

The questions on the exams are not overly complicated (no "monster" functions). The aim is to be sure you know the basics. The questions on the no calculator sections are straightforward.

Questions on the calculator allowed section are not testing differentiation skills, but rather they test if you know when to differentiate (e.g. given the velocity, what is the acceleration?) These will be reviewed in a different student session.

Try some multiple-choice questions:

1. (2003 – 1) Power and Chain rule
2. (2003 – 9) Basic functions, Chain rule and evaluate.
3. (1998 – 28) Trig function, Chain rule, evaluate.
4. (2003 – 14) Product rule, Chain rule

5. (2003 – 4) Quotient rule, Chain rule

6. (1997 – 2) Product rule with a radical.

   Hint: Radicals can always be handled by writing them as fractional exponents. However, since *square roots* occur so often, It is faster to memorize one additional rule: $d\left(\sqrt{u}\right)=\dfrac{du}{2\sqrt{u}}$, (in words: "the derivative of a square root is the derivative of the radicand divided by twice the radical.") This allows you, in one step, for this example to get to

   $$f'(x)=x\frac{2}{2\sqrt{2x-3}}+\sqrt{2x-3}(1)$$

   After this you must simplify to get one of the answer choices.

7. (2003 – 26) Implicit differentiation, evaluate.

8. (1997 – 17) Implicit differentiation second derivative.

9. (1998 – 15) FTC without Chain rule

10. (2003 – 23) FTC with Chain rule.

    An example:

    $$\begin{aligned}\int_{\frac{\pi}{6}}^{x^3}\cos(t)\,dt&=\sin(t)\Big|_{\frac{\pi}{6}}^{x^3}\\&=\sin\left(x^3\right)-\sin\left(\tfrac{\pi}{6}\right)\\&=\sin\left(x^3\right)-\frac{1}{2}\end{aligned}$$

    Differentiate this answer using the Chain rule

    $$\begin{aligned}\frac{d}{dx}\int_{\frac{\pi}{6}}^{x^3}\cos(t)\,dt&=\frac{d}{dx}\left(\sin\left(x^3\right)-\frac{1}{2}\right)\\&=\cos\left(x^3\right)\left(3x^2\right)-0\\&=3x^2\cos\left(x^3\right)\end{aligned}$$

    This can be had without all this trouble by substituting the upper limit of integration into the integrand and multiplying the result by the derivative of the upper limit of integration.

11. (2003 – BC 18 Suitable for AB)) FTC, Chain Rule, values from graph.

Student Notes

12. (2003 – BC 79 Suitable for AB) Chain rule, values from table. Of course $h(x)$ could be $f(x)g(x)$ or $\dfrac{f(x)}{g(x)}$, etc.

WATCH and **LISTEN** to the multiple-choice questions being solved.

Go to http://tinyurl.com/NMSI-Math-1 Click on the "Full Screen" arrow. Then click anywhere on the page to see and hear from that point on. Click to go back anytime.

Multiple-choice answers: 1 e, 2 a, 3 e, 4 e, 5 d, 6 a, 7 b, 8 a, 9 d, 10 e, 11 c, 12 d.

Free-response Questions

Derivatives in free-response questions are almost always used in the process of some application and will be considered in other prep sessions. Here are some free-response implicit differentiation questions.

2004 AB4/BC4 Some comments:

a. (Part a) The answer is given here because if a student makes a mistake, or can't do this part, then there's no way they can go on and do any of the other parts of the question. This is an easy 2 points.

b. (Part b) A horizontal tangent line requires the numerator of the derivative to be zero but the denominator to not be zero. Why?

c. (Part c) In calculating the numerical value of the second derivative the value of y' (= 0) must be substituted into the first derivative. This can be done numerically; it is not necessary to substitute the symbolic first derivative to determine the numerical value.

d. (Part c) having found the values of the first and second derivatives, the second derivative test is a slam dunk.

2000 AB5/BC5 is similar to the problem above.

2002 BC 5 (d). Do only part (d). Parts (a) and (c) are differential equation questions suitable for AB students (part (b) Euler's method is a BC only topic). Part (d) is another second derivative test question using implicit differentiation.

Name: ________________________ Class: ___________________ Date: __________ ID: A

Calculate the Derivative

Multiple Choice
Identify the choice that best completes the statement or answers the question.

____ 1. If $y = \left(x^3 + 1\right)^2$, then $\dfrac{dy}{dx} =$

a. $\left(3x^2\right)^2$
b. $2\left(x^3 + 1\right)$
c. $2\left(3x^2 + 1\right)$
d. $3x^2\left(x^3 + 1\right)$
e. $6x^2\left(x^3 + 1\right)$

____ 2. If $f(x) = \ln\left(x + 4 + e^{-3x}\right)$, then $f'(0)$ is

a. $-\dfrac{2}{5}$
b. $\dfrac{1}{5}$
c. $\dfrac{1}{4}$
d. $\dfrac{2}{5}$
e. nonexistent

____ 3. If $f(x) = \tan(2x)$, then $f'\left(\dfrac{\pi}{6}\right) =$

a. $\sqrt{3}$
b. $2\sqrt{3}$
c. 4
d. $4\sqrt{3}$
e. 8

____ 4. If $y = x^2 \sin 2x$, then $\frac{dy}{dx} =$

a. $2x \cos 2x$
b. $4x \cos 2x$
c. $2x(\sin 2x + \cos 2x)$
d. $2x(\sin 2x - x \cos 2x)$
e. $2x(\sin 2x + x \cos 2x)$

____ 5. If $y = \frac{2x+3}{3x+2}$, then $\frac{dy}{dx} =$

a. $\frac{12x + 13}{(3x+2)^2}$
b. $\frac{12x - 13}{(3x+2)^2}$
c. $\frac{5}{(3x+2)^2}$
d. $\frac{-5}{(3x+2)^2}$
e. $\frac{2}{3}$

____ 6. If $f(x) = x\sqrt{2x-3}$, then $f'(x) =$

a. $\frac{3x-3}{\sqrt{2x-3}}$
b. $\frac{x}{\sqrt{2x-3}}$
c. $\frac{1}{\sqrt{2x-3}}$
d. $\frac{-x+3}{\sqrt{2x-3}}$
e. $\frac{5x-6}{2\sqrt{2x-3}}$

____ 7. What is the slope of the line tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point (3, 2)?

a. 0
b. $\frac{4}{9}$
c. $\frac{7}{9}$
d. $\frac{6}{7}$
e. $\frac{5}{3}$

____ 8. If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point (4, 3) ?

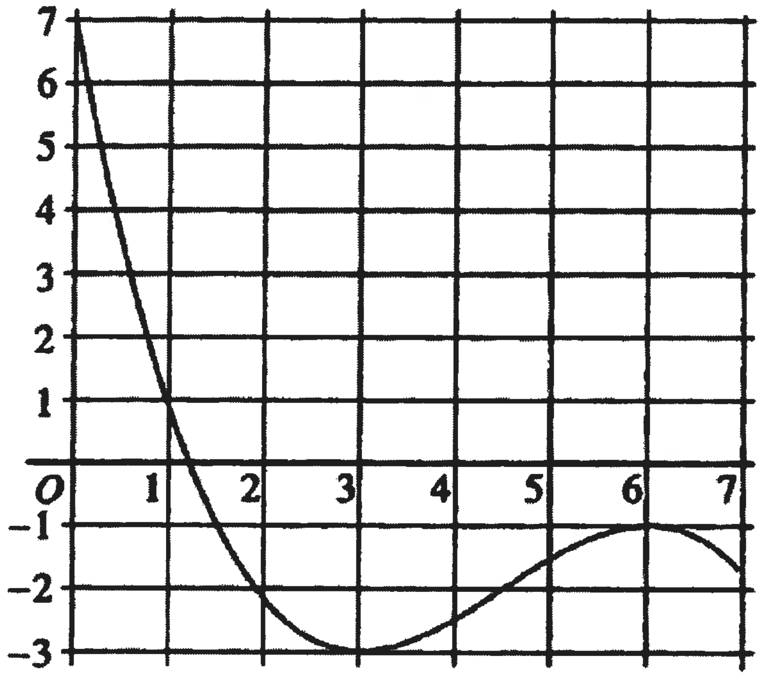
a. $-\frac{25}{27}$
b. $-\frac{7}{27}$
c. $\frac{7}{27}$
d. $\frac{3}{4}$
e. $\frac{25}{27}$

____ 9. If $F(x) = \int_0^x \sqrt{t^3 + 1}\,dt$, then $F'(2) =$

a. -3
b. -2
c. 2
d. 3
e. 18

____ 10. $\frac{d}{dx}\left(\int_0^{x^2} \sin\left(t^3\right)dt\right) =$

a. $-\cos\left(x^6\right)$
b. $\sin\left(x^3\right)$
c. $\sin\left(x^6\right)$
d. $2x\sin\left(x^3\right)$
e. $2x\sin\left(x^6\right)$



Graph of f

____ 11.

The graph of the function f shown in the figure above has horizontal tangents at $x = 3$ and $x = 6$. If $g(x) = \int_0^{2x} f(t)dt$, what is the value of $g'(3)$?

a. 0
b. -1
c. -2
d. -3
e. -6

____ 12.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	6	5	3	-2
1	3	-3	-1	2
3	1	-2	2	3

The table above gives values of f, f', g, and g' at selected values of x. If $h(x) = f(g(x))$, then $h'(1) =$

a. 5
b. 6
c. 9
d. 10
e. 12

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

   (b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

   (c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

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Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a)
$$2x + 8yy' = 3y + 3xy'$$
$$(8y - 3x)y' = 3y - 2x$$
$$y' = \frac{3y - 2x}{8y - 3x}$$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3,\ 3y = 6$
$y = 2$

$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3 : \begin{cases} 1 : \frac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$

At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(2xy - x^3) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

$4\begin{cases} 1: \ y^2 - y = 6 \\ 1: \ \text{solves for } y \\ 2: \ \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x(2y - x^2) = 0$ gives $x = 0$ or $y = \frac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \frac{1}{2}x^2$, $\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$

$-\frac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$3\begin{cases} 1: \ \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to } 0 \\ 1: \ \text{substitutes } y = \frac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad\ \text{into the equation for the curve} \\ 1: \ \text{solves for } x\text{-coordinate} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

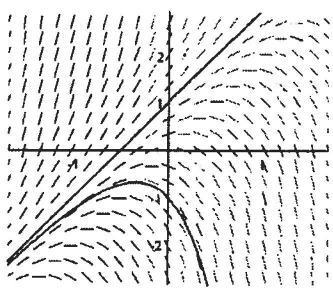
(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.

(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(a)



$$2\begin{cases} 1: \text{ solution curve through } (0,1) \\ 1: \text{ solution curve through } (0,-1) \end{cases}$$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 1 + (2 - 0)(0.1) = 1.2$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

$$2\begin{cases} 1: \text{ Euler's method equations or equivalent table applied to (at least) two iterations} \\ 1: \text{ Euler approximation to } f(0.2) \text{ (not eligible without first point)} \end{cases}$$

(c) Substitute $y = 2x + b$ in the DE:

$2 = 2(2x + b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

$$2\begin{cases} 1: \text{ uses } \frac{d}{dx}(2x + b) = 2 \text{ in DE} \\ 1: \ b = 1 \end{cases}$$

(d) g has local maximum at $(0,0)$.

$g'(0) = \left.\frac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and

$g''(x) = \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$, so

$g''(0) = 2\,g'(0) - 4 = -4 < 0$.

$$3\begin{cases} 1: \ g'(0) = 0 \\ 1: \text{ shows } g''(0) = -4 \\ 1: \text{ conclusion} \end{cases}$$